Roll No. ....

DD–2804

M. A./ M. Sc. (Previous) EXAMINATION, 2020

MATHEMATICS

Paper Fourth

(Complex Analysis)

Time : Three Hours

Maximum Marks : 100

Note : Attempt any *two* parts of each question. All questions carry equal marks.

Unit—I

1. (a) State and prove Cauchy's Integral formula.

(b) Let $f(z)$ be analytic in the region $|z| < \rho$ and let $z = re^{i\theta}$ be any point of this region. Then :

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi,$$

where R is any number such that $0 < R < \rho$.

(c) Show that :

$$e^{\frac{1}{2}c\left(z - \frac{1}{2}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta,\ c > 0$.

Unit—II

2. (a) Show that :

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \; a > b > 0.$$

(b) Cross-ratios are invariant under a bilinear transformation. Prove.

(c) Show that the transformation $w = \tan^2\left(\frac{\pi}{4}\sqrt{z}\right)$ transforms the interior of the unit circle $|w| = 1$ into the interior of a parabola.

Unit—III

3. (a) If $|z| \le 1$ and $p \ge 0$, then :

$$|1 - E_p(z)| \le |z|^{p+1},$$

where $E_p(z)$ is elementary factor.

(b) Show that the function :

$$f_1(z) = 1 + z + z^2 + z^3 + \ldots.. + z^n + \ldots...$$

can be obtained outside the circle of convergence of the power series.

(c) State and prove Harnack's inequality.

Unit—IV

4. (a) If $f(z)$ is analytic within and on the circle γ such that $|z| = R$ and if $f(z)$ has zeros at the points $a_i \ne 0$, $(i = 1, 2, 3, \ldots., m)$ and poles at $b_j \ne 0$, $(j = 1, 2, 3, \ldots., n)$ inside γ, multiple zeros and poles being repeated, then :

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{R}e^{i\theta})|\, d\theta = \log|f(0)| + \sum_{i=1}^{m} \log\frac{\mathrm{R}}{|a_i|} - \sum_{j=1}^{n} \log\frac{\mathrm{R}}{|b_j|}.$$

(b) State and prove Poisson-Jensen formula.

(c) State and prove Hadamard's three circle theorem.

Unit—V

5. (a) Let g be analytic in $B(0; R)$, $g(0) = 0$, $|g'(0)| = \mu > 0$ and $|g(z)| \le M$ for all z, then $g(B(0; R)) \supset B\left(0; \frac{R^2\mu^2}{6M}\right)$.

(b) State and prove Schottky's theorem.

(c) Let $F \in H(D - \{0\})$, F be one-to-one in D and

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n \; (z \in D),$$

then $\sum_{n=1}^{\infty} n|\alpha_n|^2 \le 1$.